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ON ABELIAN π-REGULAR RINGS

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INTRODUCTION

Throughout this paper the letter R denotes an associative ring with 1, Id(R) denotes the set of all idempotent elements of R, C(R) denotes the center of R, and Nil(R) denotes the set of all nilpotent elements of R. A π-regular ring R is called an abelian π-regular ring if Id(R) is a subset of C(R). Recall that a ring R is called strongly π-regular if for every $x \in R$ there exist $n \geq 1$ and $y \in R$ such that $x^{2n}y = x^n$. It is easy to see that an abelian π-regular ring is strongly π-regular. In [14, Theorem 2, (2)], Ohori showed that in an abelian π-regular ring R, the Nil(R) is a two-sided ideal of R and R/Nil(R) is regular. His proof relies on [1, Lemma 1] and [4, Remark]. The purpose of this paper is to give an alternative proof of this fact and in Theorem 3 we prove the converse of this fact. Also, we show that every element in an abelian π-regular ring R is a sum of two units if and only if $Z/2Z$ is not a homomorphic image of R. Recall an element x of R is called regular (unit regular) if there exists $y \in R$ (a unit u in R) such that $xyx=x$ ($xux=x$).

We start with the following lemma :

Lemma 1. Let $x \in R$. If x is unit regular, then $x=eu$ for some $e \in Id(R)$ and $u \in U(R)$, where $U(R)$ denotes the set of all units of R.

Proof. Suppose x is unit regular. Then for some $v \in U(R)$ we have $xvx=x$. Let $e=xv \in Id(R)$ and $u=v^{-1}$. Then $x=eu$ ■

The following fact is needed in the proof of Theorem 1.

Fact 1. [2, Theorem 2]. Suppose $Id(R) \subset C(R)$. Let $x \in R$. If x is regular, then x is unit regular.

The following theorem gives a characterization of all π-regular elements in a ring R such that $Id(R) \subset C(R)$.

Theorem 1. Suppose $Id(R) \subset C(R)$. Let $x \in R$. Then x is π-regular if and only if there exists $e \in Id(R)$ such that ex is regular and $(1-e)x \in Nil(R)$.

Proof. Since x is π-regular, for some $n \geq 1$, x^n is regular. Hence, by Fact 1 and Lemma 1, we have $x^n = eu$ for some $e \in Id(R)$ and $u \in U(R)$. Then $ex(x^{n-1}u^{-1})ex = (ex^n u^{-1})ex = (euu^{-1})ex = ex$. Hence, ex is regular. Now, $[(1-e)x]^n = (1-e)x^n = (1-e)eu = 0$, since $(1-e) \in C(R)$ and $x^n = eu$.

For the converse, suppose for some $e \in Id(R)$, ex is regular and $(1-e)x \in Nil(R)$. Then for some $n \geq 1$, $[(1-e)x]^n = (1-e)x^n = 0$. Hence, **(*)** $ex^n = x^n$. Since ex is regular, by Lemma 1, $ex = cu$ for some $c \in Id(R)$ and $u \in U(R)$. Hence, $(ex)^n = (cu)^n = cu^n$, since $c \in C(R)$. But $(ex)^n = ex^n = x^n$ by (*). Thus $x^n = cu^n$. Let $y = cu^{-n}$. Then $x^n y x^n = x^n$ and hence x is π-regular ■

Suppose $Id(R) \subset C(R)$ and $x \in R$ such that x is π-regular. Then by the proof of the above theorem for some $e \in Id(R)$, $v \in U(R)$ and $m \geq 1$, we have $x^m = ev$ and ex is regular. Hence, by Fact 1 and Lemma 1, $ex = cw$ for some $c \in Id(R)$ and $w \in U(R)$. In fact, $e = c$. For, $e(ex) = e(cw)$. But $e(ex) = ex = cw$. Thus, $ecw = cw$ and therefore **(**)** $ec = c$. Since $e, c \in C(R)$, we have $(ex)^m = ex^m = cw^m$. Since $x^m = ev$, $ex^m = ev = cw^m$. Hence, $e = cw^m v^{-1}$. Thus $ec = cw^m v^{-1} c = cw^m v^{-1}$, since $c \in C(R)$. Hence, $ec = e$. Since $ec = c$ by **(**)** and $ec = e$, $e = c$. Thus, $ex = ew$

In light of the above argument and Theorem 1, we have

Lemma 2. Suppose $Id(R) \subset C(R)$. Let $x \in R$ such that x is π-regular. Then for some $e \in Id(R)$ and $u \in U(R)$ we have $ex=eu$.

MAJOR RESULTS

Now, we state the first major result in this paper.

Theorem 2. Suppose R is abelian π-regular. Then Nil(R) is a two-sided ideal of R.

Proof. Let $w \in Nil(R)$ and $r \in R$. Suppose rw is not in Nil(R). By Lemma 2, there exists $e \in Id(R)$ and $u \in U(R)$ such that $erw=rew=eu$. Observe that $e \neq 0$. For, if $e=0$ then $(1-e)rw=rw \in Nil(R)$ by Theorem 1 and this contradicts the assumption that rw is not in Nil(R). Since $ew \in Nil(R)$, let n be the smallest integer such that $(ew)^n=0$. Then $n \geq 2$, since $e \neq 0$. Thus, $0 = rew(ew)^{n-1} = eu(ew)^{n-1} = u(ew)^{n-1}$. Hence, $(ew)^{n-1} = 0$, a contradiction. Thus, for any $w \in Nil(R)$ and $r \in R$, we have $rw \in Nil(R)$. A similar argument will show that for any $w \in Nil(R)$ and $r \in R$, we have $wr \in Nil(R)$. Now, let $w, z \in Nil(R)$ and suppose $w+z$ is not in Nil(R). Then, once again, there exist $c \in Id(R)$, $c \neq 0$, and $v \in U(R)$ such that $c(w+z) = cv$. Hence, $cw = cv-cz = cv(1-v^{-1}z)$. Since $-v^{-1}z \in Nil(R)$,

$1-v^{-1}z = u \in U(R)$. Thus, $cw=cvu$. But $cw \in Nil(R)$ and cvu is not in Nil(R). Hence, $w+z \in Nil(R)$. Thus, Nil(R) is a two-sided ideal of R. ■

Before stating the second major result, the following two well-known lemmas are needed.

Lemma 3. Let R be a ring with 1 and I be a two-sided nil ideal of R. If $[c] \in Id(R/I)$, then there exists $e \in Id(R)$ such that $[e]=[c]$ in R/I.

Lemma 4. Let I be a two-sided nil ideal of R, K = R/I and $u \in R$. Then $[u] \in U(K)$ if and only if $u \in U(R)$.

Theorem 3. Suppose $Id(R) \subset C(R)$. Then R is π-regular if and only if Nil(R) is a two-sided ideal of R and R/Nil(R) is regular.

Proof. Suppose R is π-regular. By Theorem 2, Nil(R) is a two-sided ideal of R. Let $[x] \in R/Nil(R)$. Then for some $y \in R$ and $n \geq 1$, $x^n y x^n = x^n$. Thus, $e = x^n y \in Id(R)$ and therefore $1-e \in Id(R)$. Since $1-e \in C(R)$, $((1-e)x)^n = (1-e)x^n = (1-x^n y)x^n = 0$. Thus, $(1-e)x = (1-x^n y)x \in Nil(R)$. Thus, $[x][x^{n-1}y][x] = [x^n y][x] = [x]$.

Suppose Nil(R) is a two-sided ideal of R and K = R/Nil(R) is regular. Let $x \in R$. By Fact 1, [x] is unit regular in K and, by Lemma 1, $[x]=[c][u]$ for some $[c] \in$

$Id(K)$ and $[u] \in U(K)$. By Lemma 3, there exists $e \in Id(R)$ such that $[c]=[e]$ and by Lemma 4, $u \in U(R)$. Thus, $x=eu+w$ for some $w \in Nil(R)$. Now, $ex = e(u+w)$. Since $w \in J(R)$, $u+w \in U(R)$, where $J(R)$ denotes the Jacobson radical of R. Thus, ex is regular. Further, $(1-e)x = x-ex = (eu+w) - (eu+ew) = w-ew \in Nil(R)$. Hence, $(1-e)x \in Nil(R)$. Thus, by Theorem 1, x is π-regular. ■

Suppose a ring R is an abelian π-regular ring. Since $Nil(R)$ is a two-sided ideal of R, $Nil(R) \subset J(R)$. Since $R/Nil(R)$ is regular by Theorem 3 and the Jacobson radical of any regular ring is 0, we have $J(R) = Nil(R)$.

Lemma 5. Suppose R is abelian π-regular. Then $J(R) = Nil(R)$.

The following result follows from Theorem 3 and Lemma 1.

Corollary 1. A ring R is abelian π-regular if and only if $Id(R) \subset C(R)$, $Nil(R)$ is a two-sided ideal of R, and for every $x \in R$ there exist $e \in Id(R)$, $u \in U(R)$, and $w \in Nil(R)$ such that $x=eu+w$.

In light of Theorems 1 and 3, we have :

Theorem 4. Suppose Id(R) is a subset of C(R). Then R is π-regular if and only if for some two-sided nil ideal I of R, K=R/I is π-regular.

Proof. Suppose R is π-regular. By Theorem 2, I = Nil(R) is a two-sided ideal of R, and by Theorem 3, K = R/I is regular and hence π-regular.

For the converse, assume that R/I is π-regular for some two-sided nil ideal I of R. Then Nil(R/I) = Nil(R)/I is a two-sided ideal of R/I by Theorem 3. So Nil(R) is a two-sided ideal of R. Since R/I is π-regular, so is R/Nil(R). Therefore by Theorem 3, R is π-regular. ■

A consequence of the above theorem is the following corollary

Corollary 2. Suppose Id(R) is a subset of C(R). Then R is π-regular if and only if R/N(R) is π-regular where N(R) is the prime radical of R.

RELATED RESULTS

Recall, a prime ideal P of a ring R is called completely prime iff R/P is domain. It is well-known that if Id(R)⊂C(R) and R is regular and I is a prime ideal of R, then R/I is a division ring. However, the above fact is not always true for an abelian π-regular ring R. The referee provided us with a counter-example, see [13, Proposition 1.11] and [3, example 3.3]. But, we are able to state the following result :

Theorem 5. Suppose R is abelian π-regular and let P be a prime ideal of R, then every element in $K = R/P$ is either a nilpotent element of K or a unit element of K. In particular, if P is a prime ideal of R containing Nil(R) (e.g., a left or right primitive ideal of R), then K is a division ring.

Proof. Let $x \in R$ such that $x \notin P$. Then for some $e \in Id(R)$ and $u \in U(R)$ and $n \geq 1$, we have $x^n = eu$ by Lemma 1. Now, if $e \in P$, then $x \in Nil(K)$. Hence, suppose that $e \notin P$. Thus, $eu \notin P$. Since $(1-e)Re \subset P$ and $e \notin P$, $(1-e) \in P$. Thus $[e]=[1]$ in R/P. Thus $[x^n]=[eu]=[u]$ in R/P. But $[x^n] = [u]$ in R/P implies $[x^n]$ is a unit in R/P and therefore [x] is a unit in R/P.

By Theorem 3, Nil(R) is a two-sided ideal and R/Nil(R) is a reduced regular ring. Thus every prime factor of R/Nil(R) is a division ring. Let P be a prime ideal of R containing Nil(R). Then $K = R/P$ is a prime factor ring of R/Nil(R) and so K is a division ring. Particularly, if P is a left (or right) primitive ideal of R, then note that Nil(R) = J(R) by Lemma 5 and so $Nil(R) \subset P$. Thus the ring K is a division ring. ■

Remark. Let K and P as in the above theorem. It is easy to see that K = R/P is a division ring iff R/P is domain iff P is completely prime.

Ehrlich [5] showed that if R is a unit regular ring, then every element in R is a sum of two units. A ring R is called an (s,2)-ring [11], see also [7], if every element in R is a sum of two units of R. The following theorem gives a characterization of all abelian π-regular (s,2)-rings.

Theorem 6. Suppose R is abelian π-regular. Then R is an (s,2)-ring if and only if Z/2Z is not a homomorphic image of R.

Proof. Suppose R is an (s,2)-ring and Z/2Z is a homomorphic image of R. Then $1 \in R$ cannot be a sum of two units. Hence, Z/2Z is not a homomorphic image of R.

Conversely, suppose Z/2Z is not a homomorphic image of R. By Theorem 5, every primitive factor of R is a division ring and hence Artinian. Thus, by [7, Theorem 2] R is an (s,2)-ring. ∎

From Theorem 6, we have the following corollaries :

Corollary 3. Let R be an abelian π-regular ring such that $2=(1+1) \in U(R)$. Then R is an (s,2)-ring.

Corollary 4. Let R be an abelian π-regular ring. Then R is an (s,2)-ring if and only if for some $d\in U(R)$, $1+d\in U(R)$.

If 2 is a nonnilpotent element in an abelian π-regular ring R, then we have

Theorem 7. Suppose R is abelian π-regular and 2 is a nonnilpotent element of R. Then there exists $e\in Id(R)$ such that $e\neq 0$, and every element in eR is a sum of two units of R.

Proof. Since 2 is π-regular, by Lemma 2 we have $e2=eu$ for some $e\in Id(R)$ and $u\in U(R)$. Since 2 is not nilpotent, we see that $e\neq 0$ and $(1-e)2$ is nilpotent by Theorem 1 and the proof of Theorem 2. Now, let $x\in eR$. By Corollary 1, there exist $c\in Id(R)$, $v\in U(R)$ and $w\in Nil(R)$ such that $x=cv+w$. Since $ex=x$, we have $x=ex= ecv+ew$. On the other hand, since $(1-e)2= 2-2e$ is nilpotent, $1-(2-2e) = -1 + 2e \in U(R)$ and so $1-2e \in U(R)$. If $c = 0$, then $1-2ec = 1 \in U(R)$. If $c \neq 0$, then $c(1-2e) = c-2ec \in U(cR) = U(cRc)$ and thus there is $a \in cR$ such that $(c-2ec)a = a(c-2ec) = c$. Therefore $(1-2ec)(a+1-c) = (c-2ec+1-c)(a+1-c) = 1$ and similarly $(a+1-c)(1-2ec) = 1$. Thus, $1-2ec \in U(R)$. Since $2e = eu$, we have $1-uec \in U(R)$. Now, $1-uec = (u^{-1} - ec)u \in U(R)$ and $u \in U(R)$. So $u^{-1} - ec \in$

$U(R)$ and hence $-u^{-1} + ec \in U(R)$. Therefore $ec = (-u^{-1} + ec) + u^{-1}$ with $-u^{-1} + ec \in U(R)$ and $u^{-1} \in U(R)$. Now for our convenience, let $z = -u^{-1} + ec$ and $d = u^{-1}$. Hence, $x = (z+d)v + ew = zv + (dv+ew)$. Since $ew \in Nil(R)$ and $Nil(R) = J(R)$, $(dv+ew) \in U(R)$. Thus, x is a sum of two units of R. ■

Observe that if 2 is a nonnilpotent element of R, then this does not imply that R is an (s,2)-ring. For example, $R = Z_6$ is abelian π-regular and 2 is a nonnilpotent element of R, but R is not an (s,2)-ring. However, $4 \in Id(R)$ and every element in 4R is a sum of two units.

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